# Energy level dynamics in systems with weakly multifractal eigenstates: equivalence to 1D correlated fermions at low temperatures

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It is shown that the parametric spectral statistics in the critical random matrix ensemble with multifractal eigenvector statistics are identical to the statistics of correlated 1D fermions at finite temperatures. For weak multifractality the effective temperature of fictitious 1D fermions is proportional to $T_{eff} \propto (1-d_n)/n \ll 1$, where $d_n$ is the fractal dimension found from the $n$-th moment of inverse participation ratio. For large energy and parameter separations the fictitious fermions are described by the Luttinger liquid model which follows from the Calogero-Sutherland model. The low-temperature asymptotic form of the two-point equal-parameter spectral correlation function is found for all energy separations and its relevance for the low temperature equal-time density correlations in the Calogero-Sutherland model is conjectured.

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The spectral statistics in complex quantum systems are signatures of the underlying dynamics of the corresponding classical counterpart. The spectral statistics in chaotic and disordered systems in the limit of infinite dimensionless conductance $g$ is described by the classical random matrix theory of Wigner and Dyson [1] (WD statistics). The WD statistics possess a remarkable property of universality: it depends only on the symmetry class with respect to the time-reversal transformation $\mathcal{T}$. The three symmetry classes correspond to the lack of $\mathcal{T}$-invariance (the unitary ensemble, $\beta = 2$); the $\mathcal{T}$-invariant systems with $\mathcal{T}^2 = 1$ (the orthogonal ensemble, $\beta = 1$), and the $\mathcal{T}$-invariant systems with $\mathcal{T}^2 = -1$ (the symplectic ensemble, $\beta = 4$), respectively. The physical ground behind this universality is the structureless eigenfunctions in the ergodic regime which implies the invariance of the eigenfunction statistics with respect to a unitary transformation of the basis.

In real disordered metals the eigenfunctions are not basis-invariant. The basis-preference reaches its extreme form for the strongly impure metals where all eigenfunctions are localized in the coordinate space but delocalized in the momentum space. In this case the spectral statistics is Poissonian in the thermodynamic (TD) limit.

For low-dimensional systems $d = 1, 2$, where all states are localized in the TD limit, one can observe the smooth crossover from the WD to the Poisson spectral statistics as a function of the parameter $\xi/L$, where $\xi$ is the localization radius and $L$ is the system size. The dependence of the spectral correlation functions on the energy variable $s = E/\Delta$ ($\Delta$ is the mean level separation) is non-universal for finite $L/\xi$ but all of them tend to the Poisson limit as $L/\xi \to \infty$.

In systems of higher dimensionality $d > 2$ the situation is different because of the presence of the Anderson localization transition at a critical disorder $W = W_c$. In the metal phase $W < W_c$ the dimensional conductance $g(L) \to \infty$ as $L \to \infty$ and one obtains the WD spectral statistics in the TD limit. In the insulator state $g(L) \to 0$ at $L \to \infty$ and the limiting statistics is Poissonian. However, there is a fixed point $W = W_c$ in which the spectral statistics are nearly independent of $L$. Thus at the critical point there exist universal spectral statistics which are neither WD nor Poissonian but rather a hybrid of both [2]. However, the universality of the critical spectral statistics (CSS) is somewhat limited, since it depends not only on the Dyson symmetry parameter $\beta$ but also on the critical value of the dimensional conductance $g^*$ which in turn depends on the dimensionality $d$ of the system [3]. Thus for each universality class there is a *family* of critical spectral statistics parametrized by the critical dimensionless conductance $g^*$.

The very existence of the subject of critical level statistics imposes a constraint on the possible values of the localization length exponent $\nu = [d\beta(g)/d\ln g]^{-1}|_{g=g^*}$, where $\beta(g) \equiv d\ln g/d\ln L$ is the scaling function. Indeed, for the spectral statistics to be meaningful the width of the critical energy window $\delta E$ must be much larger than the mean level separation $\Delta \propto 1/L^d$. The quantity $\delta E$ is defined as the distance from the mobility edge $E = E_c$ at which the localization or correlation radius $\xi(\delta E) \propto |\delta E|^{-\nu}$ is equal to the system size $L$. The number of critical eigenstates $\mathcal{N} = \delta E/\Delta$ is proportional to $L^{d-\frac{1}{\nu}}$. For $\nu d > 1$ this number tends to infinity in the TD limit $L \to \infty$ despite the width of the critical energy window shrinks to zero. This necessary condition for the existence of the critical statistics is secured by the famous Harris criterion $\nu d > 2$.

However, the critical exponent $\nu$ enters not only in the necessary condition for the CSS but also in the correlation functions of the density of energy levels $\rho(E)$. It has been shown in [4,5] that there is a power-law tail in the critical two-level correlation function (TLCF) $R(\omega) = \langle\langle \rho(E)\rho(E+\omega)\rangle\rangle$ that arises because of the finite-size correction to the dimensionless conductance $g(L_\omega)/g^* - 1 \propto (L_\omega/L)^{1/\nu} = s^{-\frac{1}{\nu d}}$, where $L_\omega \propto \omega^{-1/d} \ll L$ is the length scale set by the energy difference $\omega = E - E' \equiv s\Delta$ between two levels. The sign of this tail depends on whether the critical energy levels are on the metal ($E < E_c$) or on the insulator ($E > E_c$) side of the mobility edge, in the same way as for the 2D systems in the weak delocalization ($\beta = 4$) or weak localization ($\beta = 1$) regimes [12]. Clearly, this power-law tail does not reflect properties of the critical eigenstates but rather the behavior $\xi(|E-E_c|)$ of the size of the space region where eigenstates show the critical space correlations.

In order to study the relationship between the properties of the critical eigenstates and the CSS in its pure form one should consider a system with a continuous line of critical points where $\beta(g) = 0$. This case formally corresponds to $\nu = \infty$ and the finite-size effects are absent.

Another complication which makes ambiguous the definition of CSS is the fact that being independent of the system size, the spectral correlation functions depend on the boundary conditions [13–15] and topology of a system. Therefore we will consider the system of the torus topology where CSS takes its 'canonical' form. In particular, the TLCF decays exponentially in this case.

As has been already mentioned the universality of the WD statistics is based on the ergodic, basis-invariant statistics of eigenfunctions which one may encounter in different physical situations. The characteristic feature of all critical quantum systems is the multifractal statistics of the critical eigenfunctions [6–8]. The simplest two-point correlations of the critical wave functions can be obtained from the renormalization group result [10] for $l < r < \xi < L$:

$$\langle |\Psi_E(0)|^{2n} |\Psi_E(r)|^{2n}\rangle = p < |\Psi_E(0)|^{2n} >^2 (\xi/r)^{\alpha_n}, \quad (1)$$

where $l$ is the short-distance cut-off of the order of the elastic scattering length, $p = (\xi/L)^d$ is the probability for a reference point to be inside a localization region. The exponent $\alpha_n = 2n(d_n - d_{2n}) + d_{2n} - 2d_n + d$ is expressed through the fractal dimensions $d_n$ defined by the $L$-dependence of the moments of inverse participation ratio:

$$L^d \langle |\Psi_E(r)|^{2n}\rangle \propto L^{-d_n (n-1)}, \qquad n \geq 2. \quad (2)$$

At the critical point $p = 1$ and the correlation radius $\xi \to \infty$ in Eq.(1) must be replaced by the sample size $L$. Spectral statistics are related with eigenfunction correlations at different energies $E$ and $E' = E + \omega$. If $\omega \gg \Delta$ and thus $L_\omega \ll L$ one should substitute $L_\omega$ for $L$ in the $r$-dependent term of Eq.(1). In this way the multifractality exponents $d_n$ enter the spectral $\omega$-dependences [9].

For weak multifractality one can expect the fractal dimensions to be a linear function of $n$ which is controlled by only one parameter $a$:

$$d_n/d = 1 - an, \qquad a \sim 1/g^* \ll 1. \quad (3)$$

This relationship holds approximately for the Anderson transition in $2 + \epsilon$ dimensions [6–8] and is fulfilled exactly for the critical eigenstate of the Dirac equation in the random vector-potential [11].

Thus for critical quantum systems with weak multifractality it is natural to expect that the spectral statistics depends on only one system-specific parameter - the critical conductance $g^*$.

In view of the expected universality, it is useful to find a simple one-parameter random matrix ensemble with the multifractal eigenfunction statistics which would play the same role for the critical systems as the classical RMT does for the ergodic systems. As a matter of fact there are few candidates [16]. However, here we focus only on one of them [17], since for this ensemble the multifractality of eigenstates has been rigorously proven [17,18].

Consider a Hermitean $N \times N$ matrix with the real ($\beta = 1$), complex ($\beta = 2$) or quaternionic ($\beta = 4$) entries $H_{ij}$ ($i \geq j$) which are independent Gaussian random numbers with zero mean and the variance:

$$\langle |H_{ij}|^2\rangle = \frac{1}{1 + \frac{(i-j)^2}{B^2}} \begin{cases} 1/\beta, & i = j, \\ 1/2, & i \neq j \end{cases} \equiv J(i-j) \quad (4)$$

This model has been shown to be critical both for large [17] and for small [18] values of $B$ with the fractal dimensionality $d_2$ at the center of the spectral band being:

$$d_2 = \begin{cases} 1 - \frac{1}{\pi\beta B}, & B \gg 1, \\ 2B, & B \ll 1 \end{cases} \quad (5)$$

Thus the 1D system with long-range hopping described by the matrix Hamiltonian Eq.(4) possesses the line of critical points $B \in (0, \infty)$, the fractal dimensionality $d_2$ changing from 1 to 0 with decreasing $B$.

One can extend this matrix 1D model by closing it into a ring and applying a 'flux' $\varphi \in [0, 1]$. In this case

$$H_{ij}(\varphi) = H_{ij} + H_{ij}^{(1)}\, e^{2\pi i\varphi\, sgn(i-j)} \quad (6)$$

is a sum of two independent Gaussian random numbers with the variance of $H_{ij}^{(1)}$ given by:

$$\langle |H_{ij}^{(1)}|^2\rangle = J(N - |i-j|). \quad (7)$$

For large values of $B$ which correspond to weak multifractality one can derive [17] an effective field theory – the supersymmetric nonlinear sigma-model [19] – which describes the spectral and eigenfunction correlations of the critical random matrix ensemble Eq.(4):

$$F[\mathbf{Q}] = -\frac{g^*}{16}\sum_{i,j=1}^{N} Str\,[\mathbf{Q}_i\,U_{|i-j|}\,\mathbf{Q}_j] + \frac{i\pi s}{4N}\sum_{i=1}^{N} Str[\sigma_z \mathbf{Q}_i], \tag{8}$$

where $\mathbf{Q}$ is the supermatrix with $\mathbf{Q}_i^2 = \mathbf{1}$ and

$$g^* = 4\beta B. \tag{9}$$

The symmetry with respect to time reversal is encoded in the symmetry of $\mathbf{Q}_i$ in exactly the same way as for the diffusive sigma-model [19]. The only difference is the long-range kernel $U_{|i-j|}$ with the Fourier-transform $\tilde{U}_k = |k|$. For a torus geometry $k = 2\pi m/N$, where $m$ is an *arbitrary* integer.

One can explicitly resolve the constraint $\mathbf{Q}^2 = 1$ by switching to the integration over the 'angles' $W$. Then the Gaussian fluctuations of 'angles' recover the spectrum of 'quasi-diffusion' modes:

$$\varepsilon_m = g^*\,|m|, \qquad m = 0, \pm 1, \pm 2, ... \tag{10}$$

The problem of spectral statistics can be generalized to include the dependence of spectrum on the flux $\varphi$ introduced by Eq.(6). One can define [20] the *parametric* two-level correlation function $R(s,\varphi) = \langle\langle \rho(E,0)\rho(E + s\Delta,\varphi)\rangle\rangle$ which can be treated in the framework of the same nonlinear sigma-model but with the phase-dependent $\varepsilon_m$:

$$\varepsilon_m(\varphi) = g^*\,|m-\varphi|. \tag{11}$$

Following the work by Andreev and Altshuler Ref. [21] we introduce the spectral determinant:

$$D^{-1}(s,\varphi) = \prod_{m\neq 0}\frac{\varepsilon_m^2(\varphi) + s^2}{\varepsilon_m^2(0)} \tag{12}$$

Then it can be shown in the same way as in Ref. [21] that the parametric TLCF for $s \gg 1$ and $g^*\varphi \gg 1$ can be expressed in terms of the spectral determinant as follows:

$$R^u(s,\varphi) = -\frac{1}{4\pi^2}\frac{\partial^2 G(s,\varphi)}{\partial s^2} + \cos(2\pi s)\,e^{G(s,\varphi)} \tag{13}$$

$$R^o(s,\varphi) = -\frac{1}{2\pi^2}\frac{\partial^2 G(s,\varphi)}{\partial s^2} + 2\cos(2\pi s)\,e^{2G(s,\varphi)} \tag{14}$$

$$R^s(s,\varphi) = -\frac{1}{8\pi^2}\frac{\partial^2 G(s,\varphi)}{\partial s^2} + \frac{\pi}{\sqrt{8}}\,\cos(2\pi s)\,e^{G(s,\varphi)/2} + \\ + \frac{1}{8}\,\cos(4\pi s)\,e^{2G(s,\varphi)}, \tag{15}$$

where $G(s,\varphi) = G(s,\varphi+1)$ is a periodic in $\varphi$ function:

$$e^{G(s,\varphi)} = \frac{D(s,\varphi)}{2\pi^2\,(s^2 + \varepsilon_0^2(\varphi))}. \tag{16}$$

Eqs.(13-15) coincide with the corresponding formulae in Ref. [21] for the unitary, orthogonal and symplectic ensembles after some misprints are corrected as in Ref. [22] and $s \to 2s$ for the symplectic ensemble to take account of the Kramers degeneracy. The only difference is in the form of the spectral determinant Eq.(12) due to the specific spectrum $\varepsilon_m(\varphi)$ of the quasi-diffusion modes.

Using the functional representation Eq.(8) one can also find the leading term in the deviation from the WD statistics at $s \ll g^*$ and $\varphi = 0$ using the results of Ref. [23,17]:

$$\delta R(s) = \frac{1}{2\pi^2\beta}\left(\sum_{m\neq 0}\frac{1}{\varepsilon_m^2}\right)\frac{d^2}{ds^2}\left[s^2\,R_{WD}(s)\right], \tag{17}$$

where $R_{WD}(s)$ is the Wigner-Dyson TLCF.

A remarkable property of the function $G(s,\varphi)$ for the critical sigma-model Eq.(8) on a torus is that it can be decomposed into the sum $G(s,\varphi) = F(z) + F(\bar{z})$ of analytic functions $F(z)$ and $F(\bar{z})$ where $\tau = g^*\varphi$, $z = \tau + is$, $\bar{z} = \tau - is$ and

$$F(z) = -\ln[\sqrt{2}g^*\,\sin(\pi z/g^*)]. \tag{18}$$

Eq.(18) results from a straightforward evaluation [16] of the product in Eq.(12).

On the other hand, it can be easily verified that $G(s,\varphi)$ given by Eq.(18) is proportional to the Green's function of the free-boson field $\Phi(z)$ on a torus in the $(1+1)$ $z$-space: $0 < \Re z < g^*$, $-\infty < \Im z < +\infty$:

$$\langle \Phi(s, g^*\varphi)\,\Phi(0,0)\rangle_S - \langle \Phi(0,0)\,\Phi(0,0)\rangle_S = K\,G(s,\varphi), \tag{19}$$

where $\langle ... \rangle_S$ denotes the functional average with the free-boson action:

$$S[\Phi] = \frac{1}{8\pi K}\int_0^{g^*} d\tau \int_{-\infty}^{+\infty} ds\;\left[(\partial_s\Phi)^2 + (\partial_\tau\Phi)^2\right]. \tag{20}$$

Now we are in a position to make a crucial step and suggest that for the critical RMT described by Eq.(4), the Andreev-Altshuler equations (13-15) are nothing but density-density correlations in the Luttinger liquid of fictitious 1D fermions at a finite temperature $T = 1/g^*$:

$$R(s,\varphi) = \bar{n}^{-2}\,\langle n(s,\tau)\,n(0,0)\rangle_S\, - 1, \quad \tau = g^*\varphi. \tag{21}$$

Indeed, the density operator $n(s,\tau)$ ($s$- is space and $\tau \in (0, 1/T)$ is imaginary time coordinate) for 1D interacting fermions with the Fermi-momentum $k_F = \pi$ can be expressed through the free boson field $\Phi(s,\tau)$ as follows [24]:

$$n(s,\tau) = \frac{1}{2\pi}\,\partial_s\Phi(s,\tau) + A_K\,\cos[2\pi s + \Phi(s,\tau)] + \\ + B_K\,\cos[4\pi s + 2\Phi(s,\tau)]. \tag{22}$$

The constants $A_K$ and $B_K$ are independent of 'temperature' $1/g^*$ but depend on the interaction constant $K$.

They can be uniquely determined from the WD limit $g^* \to \infty$.

Using Eqs.(21,22) and the well known result for the Gaussian average of the exponent:

$$\langle e^{ip\Phi(s,\tau)}\, e^{-ip\Phi(0,0)}\rangle = e^{Kp^2\,[G(s,\tau)-G(0,0)]}, \qquad (23)$$

one can verify that for the choice:

$$K = \frac{2}{\beta}, \qquad \beta = 1, 2, 4 \qquad (24)$$

the Andreev-Altshuler formulae Eqs.(13-15) are reproduced exactly for the orthogonal, unitary and symplectic ensembles, respectively. Now we remind on a known result that the parametric spectral statistics in the WD limit $g^* = \infty$ is equivalent to the Tomonaga-Luttinger liquid at zero temperature. It follows directly from Ref. [25] and the equivalence of the Calogero-Sutherland model and the Tomonaga-Luttinger liquid for large distances $|z| \gg 1$. The critical random matrix ensemble Eq.(4,6,7) and the critical 1D sigma-model Eq.(8) turns out to be the simplest *generalization* of the WD theory that retains the Tomonaga- Luttinger liquid analogy extended for finite 'temperatures' $T = 1/g^*$ which are related with the spectrum of fractal dimensions Eq.(3).

The Wigner-Dyson two-level statistics for all three symmetry classes can be expressed through the single kernel $K(s) = \sin(\pi s)/(\pi s)$ in the following way [1]:

$$R^u(s) = -K^2(s), \qquad (25)$$

$$R^o(s) = -K^2(s) - \frac{dK(s)}{ds}\int_s^\infty K(x)\,dx, \qquad (26)$$

$$R^s(s) = -K^2(s) + \frac{dK(s)}{ds}\int_0^s K(x)\,dx. \qquad (27)$$

It turns out that such a representation is also valid for the critical TLCF at $g^* \gg 1$ if the kernel is replaced by:

$$K(s) = \frac{T}{\alpha}\,\frac{\sin(\pi\alpha s)}{\sinh(\pi T s)}, \quad T = \frac{1}{g^*}. \qquad (28)$$

where $\alpha = 1$ for the orthogonal and the unitary ensemble and $\alpha = 2$ for the symplectic ensemble. The form of the kernel Eq.(28) can be guessed from the well known density correlation function for the case of free fermions in one dimension at a finite temperature that corresponds to the unitary ensemble [16,25,27]. In order to prove this statement we note that for $s \gg 1$ and $g^* \gg 1$ Eqs.(25-27) with the kernel Eq.(28) give the same leading terms as Eqs.(13-15) with $G(s,0)$ given by Eq.(18). For $s \ll g^*$ Eqs.(25-27) give corrections to the WD statistics that coincide with Eq.(17). Thus the representation Eqs.(25-27) with the kernel Eq.(28) are the correct asymptotic expressions for *both* $s \gg 1$ and $s \ll g^*$. At $g^* \gg 1$ these regions have a parametrically large overlap so that Eqs.(25-27) are valid for all values of $s$. Given that for all $s$ at $T = 0$ and for large energy separations $s \gg 1$ at $T \ll 1$ expressions Eqs.(25-27) correspond to equal-time correlations in the Calogero-Sutherland model, one can expect Eqs.(25-27) with the kernel Eq.(28) to describe the low-temperature equal-time correlations in the Calogero-Sutherland model [26] for for all $s$ and the interaction constant $K = 2/\beta$.

One can use the kernel Eq.(28) to compute the spacing distribution function $P(s, g^*)$ which can be compared with the corresponding distribution function $P_c(s)$ obtained by the numerical diagonalization of the three-dimensional Anderson model at the critical point. Such a comparison is done in Ref. [28] for $\beta = 1, 2, 4$. It turns out that identifying the parameter $g^*$ from the fitting $P(s, g^*) \sim e^{-\kappa(g^*)s}$ with the far exponential tail of $P_c(s) \sim e^{-\kappa s}$ one reproduces the entire distribution function $P_c(s)$ extremely well.

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